κ-Minkowski representations on Hilbert spaces

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Abstract

The algebra of functions on κ-Minkowski noncommutative spacetime is studied as algebra of operators on Hilbert spaces. The representations of this algebra are constructed and classified. This new approach leads to a natural construction of integration in κ-Minkowski spacetime in terms of the usual trace of operators.

1 Introduction

Some approaches to Quantum Gravity [1, 2] are based on the idea that at Planck-length distances geometry can have a form quite different from the one we are familiar with at large scales. In particular, Doplicher et al. [1] explored the possibility that Quantum Gravity corrections can be described algebraically by replacing the traditional (Minkowski) spacetime coordinates x_μ with *Hermitian operators* $\hat{\mathbf{x}}_\mu$ $(\mu, \nu = 0, 1, 2, 3)$ which satisfy nontrivial commutation relations

$$[\hat{\mathbf{x}}_\mu, \hat{\mathbf{x}}_\nu] = i\theta_{\mu\nu}(\hat{\mathbf{x}}).$$

A noncommutative spacetime of this type embodies an impossibility to fully know the short distance structure of spacetime, in the same way that in the phase space of the ordinary *Quantum Mechanics* there is a limit on the localization of a particle.

However, the idea of the spacetime noncommutativity has a more ancient origin which goes back to Heisenberg and was published for the fist time by Snyder [3, 4]. The motivation at that time was the hope that noncommutativity among coordinates could improve the singularity of quantum field theory at short-distances [5].

Noncommutative geometry emerges also at the level of effective theories, for example in the description of Strings in the presence of external fields [6, 7], or in the description of electronic systems in the presence of external magnetic-background [8].

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There is a wide literature on the simplest "canonical" noncommutativity characterized by a constant value of the commutators

$$[\hat{\mathbf{x}}_\mu, \hat{\mathbf{x}}_\nu] = i\theta_{\mu\nu},$$

where $\theta_{\mu\nu}$ is a matrix of dimensionful parameters. This noncommutative spacetime arises in the description of string M-theory in presence of external fields [9].

In this paper we consider another much studied noncommutative spacetime, called κ-Minkowski spacetime, characterized by the commutation relations

$$[\hat{\mathbf{x}}_0, \hat{\mathbf{x}}_j] = i\lambda\hat{\mathbf{x}}_j \quad [\hat{\mathbf{x}}_j, \hat{\mathbf{x}}_k] = 0, \quad j = 1, 2, 3$$

where $\lambda \in \mathbb{R}_{\setminus 0}$ represents the noncommutativity parameter[2]. To be simple we shall consider $\lambda > 0$ in this paper. This type of noncommutativity, introduced in [10], is an example of Lie-algebra noncommutativity where the commutation relations among spacetime coordinates exhibit a linear dependence on the spacetime coordinates themselves

$$[\hat{\mathbf{x}}_\mu, \hat{\mathbf{x}}_\nu] = i\zeta^\rho_{\mu\nu}\hat{\mathbf{x}}_\rho,$$

with coordinate-independent $\zeta^\rho_{\mu\nu}$. This algebra was proposed in the framework of the Planck scale Physics [11] as a natural candidate for a quantized spacetime in the zero-curvature limit.

Recently, κ-Minkowski gained remarkable attention due to the fact that it provides an example of noncommutative spacetime in which Lorentz symmetries are preserved as deformed (quantum) symmetries [12, 13, 14]. In particular, Majid&Ruegg [10] characterized the symmetries of κ-Minkowski with the so called κ-Poincaré algebra [15] already known in the Quantum Group literature as significant example of deformation of the Poincaré algebra.

The analysis of the physical implications of the deformed κ-Poincaré algebra has led to interesting hypotheses about the possibility that in κ-Minkowski particles are submitted to modified dispersion relations [16] which agree with the postulates of Doubly Special Relativity theories [17, 18], recent relativistic theories with both an observer-independent velocity scale and an observer-independent length scale (possibly given by the Planck length).

Over the past few years there has been a growing interest in the construction of field theories on κ-Minkowski (see for example [13, 19]). The approach to this study has essentially been based on the introduction of a deformation of the product among the coordinate functions. This deformed product (called *star-product*) replaces the commutative product and makes it possible to map a noncommutative theory into a theory with commutative functions multiplied through the deformed product [20]. In this way an analysis of field theories in noncommutative spacetime results to be quite similar to the one adopted in the ordinary commutative spaces.

However, several technical difficulties are encountered in this construction, and the results obtained so far are still partial, especially in comparison with the result obtained for the canonical noncommutative spacetime [21].

[2]Historically, the noncommutative parameter $\kappa = \lambda^{-1}$ was introduced. This explains the origin of the name "κ-Minkowski".

In this paper we propose a new approach to the study of κ-Minkowski spacetime based on the analysis of κ-Minkowski algebra as algebra of operators represented on Hilbert space. Besides the interest this analysis provides by itself, it might also prove useful in understanding the source of the technical problems encountered so far in the construction of a field theory on κ-Minkowski.

We write the fields in κ-Minkowski as Fourier expansion in *plane waves*, as we usually do in commutative field theory, with the difference that in this case the plane waves are functions of noncommutative coordinates. These waves are shown to be the elements of a unitary Lie group corresponding to the κ-Minkowski Lie algebra. In this way the problem of the representations of the κ-Minkowski fields reduces to the problem of the representation of a Lie group.

We show that it is possible to obtain a Schrödinger representation of this group on $L^2(\mathbb{R})$ by introducing Jordan-Schwinger (JS) maps [22] between the Quantum Mechanics operators of position and momentum and the generators of κ-Minkowski algebra.

The Schrödinger representation is the simplest one but is not the only one possible. We study the problem of the existence and the classification of the other representations of κ-Minkowski group.

Since κ-Minkowski group is a semidirect-product group, by using the technique of induced representations, we are able to classify all unitary irreducible representations (UIRs) of it. We discover that they are all unitarity equivalent to two classes of Schrödinger representations.

As the κ-Minkowski fields can be represented as operators on Hilbert space they form a C^*-algebra and we show that, under some hypotheses, this is the C^*-algebra of compact operators on $L^2(R, d\mu)$, with a particular choice of nontrivial measure $d\mu$.

Moreover the knowledge of representations of κ-Minkowski algebra leads to a natural construction of integration in κ-Minkowski as trace of operators. We show that in this way we recover a proposal of cyclic integration recently obtained in literature with different approaches.

The paper is organized as follows. In Sec. 2 we introduce the algebra of κ-Minkowski spacetime. In Sec. 3 we construct the Lie group corresponding to the κ-Minkowski algebra. Section 4 is devoted to the representation theory. In the first part of the section we use the JS maps to obtain the Schrödinger representation (on $L^2(R)$) of the κ-Minkowski group. In the second part of the section the technique of the induced representations allows us to prove that all IURs of the κ-Minkowski group are unitarily equivalent to the Schrödinger one. In Sec. 5 we discuss the properties of the κ-Minkowski C^*-algebra. Finally, in Sec. 6 we construct an integral in κ-Minkowski using the trace of operators.

2 κ-Minkowski Spacetime

Let us consider the four-dimensional Minkowski spacetime $\mathbb{R}^{(3,1)}$. It can be viewed as a vector space with basis given by the *real* coordinates x_μ[3]. This vector space can be made into a (Abelian) Lie algebra $\mathfrak{g}_0$ by introducing a trivial bilinear operation (Lie bracket) $[,]: \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathfrak{g}_0$

$$[x_\mu, x_\nu] = 0. \tag{1}$$

Now let us consider the flat vector space $\mathbb{R}^{(3,1)}$ with basis $\hat{\mathbf{x}}_\mu$ and replace the trivial Lie bracket (1) with the following one

$$[\hat{\mathbf{x}}_0, \hat{\mathbf{x}}_j] = i\lambda\hat{\mathbf{x}}_j \qquad [\hat{\mathbf{x}}_j, \hat{\mathbf{x}}_k] = 0, \qquad j,k = 1,2,3. \tag{2}$$

The flat vector space $\mathbb{R}^{(3,1)}$ endowed with this Lie bracket is called κ-Minkowski spacetime. The κ-Minkowski coordinates $\hat{\mathbf{x}}_\mu$ with relations (2) generate the κ-Minkowski Lie algebra $\mathfrak{g}_\lambda$. In the limit $\lambda \to 0$ the commutative Minkowski algebra is recovered

$$\lim_{\lambda\to 0} \mathfrak{g}_\lambda = \mathfrak{g}_0 \tag{3}$$

As we have mentioned in the Introduction, we are interested in the extension of the notion of classical fields[4] on κ-Minkowski noncommutative spacetime.

It is well known that a classical field on the commutative Minkowski spacetime $\mathbb{R}^{(3,1)}$ can be represented as a Fourier expansion in plane waves

$$f(x) = \int d^3k dw\, \tilde{f}(w,k)\, e^{iwx_0 - ik_jx_j}, \tag{4}$$

where $\tilde{f}(w,k)$ is the standard Fourier transform of the function $f(x)$, and the function $e^{iwx_0 - ik_jx_j}$ has the meaning of a plane wave.

Inspired by the commutative case we would like to write the noncommutative fields in κ-Minkowski as a Fourier expansion in noncommutative exponential functions. But because of the noncommutativity of the coordinates $\hat{\mathbf{x}}_\mu$ there is an ambiguity in generalizing the exponential function $e^{i(wx_0 - k_jx_j)}$ to the noncommutative case. One possibility is the time-to-the-right exponential

$$\hat{\mathbf{x}}_\mu \to e^{-ik_j\hat{\mathbf{x}}_j} e^{iw\hat{\mathbf{x}}_0} \tag{5}$$

but the alternative choice $e^{ik_\mu\hat{\mathbf{x}}^\mu}$ and many others are also possible. However, since all these maps are connected by a change of variables k_μ (see [12]) the particular choice of map is not relevant for our study. In this paper we prefer to base our formalism on the time-to-the right notation only because it involves some advantages in the calculations.

[3] In this paper Greek indices take values from zero to three while Latin indices take values from one to three. The Minkowski metric is taken to be of signature (+ - - -).

[4] Even though we are dealing with classical fields a sort of quantization naturally arises from the noncommutativity of the spacetime coordinates, but this quantization has not the meaning of the second-quantization in particle Physics.

Using the time-to-the right exponential the κ-Minkowski generalization of (4) takes the form

$$F(\hat{\mathbf{x}}) \equiv \mathcal{W}(f(x)) = \int d^3k\, dw\; \tilde{f}(w,k)\, e^{-ik_j\hat{\mathbf{x}}_j} e^{iw\hat{\mathbf{x}}_0} \tag{6}$$

where $\tilde{f}(w,k)$ is the classical Fourier transform of the commutative function $f(x)$. The commutative case suggests the interpretation of the noncommutative functions $e^{-ik_j\hat{\mathbf{x}}_j} e^{iw\hat{\mathbf{x}}_0}$ as noncommutative plane waves.

This "quantization" is conceptually similar to the Quantum Mechanics quantization (see [23]) for a detailed description). The basic observables of Quantum Mechanics (Q_j, P_j) satisfy the Heisenberg algebra (or canonical) commutation relations

$$[Q_i, P_j] = i\hbar \quad [Q_i, Q_j] = [P_i, P_j] = 0, \quad i, j = 1, 2, ...n \tag{7}$$

and the Weyl quantization of the Phase Space is given by the following map

$$F(Q,P) \equiv W(f) = \int d^n\alpha d^n\beta\, \tilde{f}(\alpha,\beta)\, e^{i\alpha_j Q_j + i\beta_j P_j} \tag{8}$$

which has exactly the same structure of the map in Eq. (6). The operators $W(f)$ are called Weyl operators. In analogy with this name we call the operators $\mathcal{W}(f)$ κ-Weyl operators.

The structure of the algebra of Weyl operators (8) has been largely investigated in literature. It is a C^*-algebra of the compact operators on $L^2(R)$, and its representations on Hilbert space are very well-known.

The algebra of κ-Weyl operators (6) instead is quite new and there are no studies about its representations on Hilbert spaces. The purpose of the present paper is to investigate this problem. Since the κ-Weyl operators are Fourier expansions in the elements $e^{-ik_j\hat{\mathbf{x}}_j} e^{iw\hat{\mathbf{x}}_0}$ we are essentially interested in the representations of these elements. As we will show in the next Section they are the elements of a unitary Lie group associated with the κ-Minkowski Lie algebra $\mathfrak{g}_\lambda$. Thus we are essentially interested in representing this group.

3 κ-Minkowski Lie Group

In this section the Lie groups of the κ-Minkowski Lie algebra $\mathfrak{g}_\lambda$ are constructed. The technique we use is based on the Baker-Cambpell-Hausdorff (BCH) formula and represents a generalization of the technique employed in the construction of the Heisenberg group (7). We start this section by reviewing the construction of the Heisenberg group and then we generalize the procedure to the case of κ-Minkowski.

3.1 The Heisenberg Group

The Heisenberg algebra (7) can be reformulated as a Lie algebra by introducing a further basis vector C such that

$$[Q_j, Q_k] = [P_j, P_k] = [Q_j, C] = [P_j, C] = 0, \quad [Q_j, P_k] = i\delta_{jk}C \tag{9}$$

the relation to the previous commutation relations (7) being that they correspond to the case C acts by $\hbar$. Introducing the generators $T(\alpha, \beta, \gamma) = \sum_j (\alpha_j Q_j + \beta_j P_j + \gamma C)$, Eq. (9) reads

$$[T(\alpha, \beta, \gamma), T(\alpha', \beta' \gamma')] = T(0, 0, i(\alpha\beta' - \alpha'\beta)$$

A Lie group associated with the Heisenberg algebra can be obtained using the BCH formula (see [24])

$$e^{T(\alpha,\beta,\gamma)}\, e^{T(\alpha',\beta'\gamma')} = e^{T(\alpha,\beta\gamma)+T(\alpha',\beta'\gamma')+\frac{1}{2}T(0,0,i(\alpha\beta'-\alpha'\beta))} = e^{T(\alpha+\alpha',\beta+\beta'+\gamma+\gamma'+\frac{i}{2}(\alpha\beta'-\alpha'\beta))}$$

The identification of the element $\alpha, \beta, \gamma \in \mathbb{R}^{2n+1}$ with the element $e^{T(\alpha,\beta,\gamma)}$ makes $\mathbb{R}^{2n+1}$ into a group, called Heisenberg group, with composition law

$$(\alpha, \beta, \gamma)(\alpha', \beta', \gamma') = (\alpha + \alpha', \beta + \beta', \gamma + \gamma' + \frac{i}{2}(\alpha\beta' - \alpha'\beta)) \tag{10}$$

We notice that the elements $e^{i\sum_j \alpha_j Q_j + \beta_j P_j}$ which appear in the Weyl-quantization map (8) are elements of a unitary Heisenberg group with $\gamma = 0$.

3.2 The κ-Minkowski Group

In Ref. [25] we have shown that the construction of the Heisenberg group (10) can be generalized to any Lie-algebra. The only difference is that the composition law is no longer Abelian.

Here we show this result in the case of κ-Minkowski using the same procedure adopted in the Heisenberg case above. It is sufficient to work in 1+1-dimensions, the result being straightforwardly extendable to any dimension.

Let us write the commutation relations of 1+1-dimensional κ-Minkowski algebra in the form

$$[(\alpha\hat{\mathbf{x}}_0 + \beta\hat{\mathbf{x}}), (\alpha'\hat{\mathbf{x}}_0 + \beta'\hat{\mathbf{x}})] = i\lambda(\alpha\beta' - \alpha'\beta)\hat{\mathbf{x}}$$

By introducing the generators $T = \alpha\hat{\mathbf{x}}_0 + \beta\hat{\mathbf{x}}$ we see that

$$[T(\alpha, \beta), T(\alpha', \beta')] = i\lambda T(0, \alpha\beta' - \alpha'\beta).$$

As in the case of the Heisenberg algebra, a Lie group associated with the κ-Minkowski algebra can be obtained by the BCH formula

$$e^T\, e^{T'} = e^{T+T'+\frac{1}{2}[T,T']+\frac{1}{12}[T,[T,T']+[[T,T'],T']]+..},$$

where $T = T(\alpha, \beta)$ and $T' = T(\alpha', \beta')$. The series in the exponential function at the right side has been computed [26] and the following relation has been obtained

$$e^{T(\alpha,\beta)} e^{T(\alpha',\beta')} = e^{T(\alpha+\alpha',\, \frac{\beta\phi(\alpha)+e^{i\lambda\alpha}\beta'\phi(\alpha')}{\phi(\alpha+\alpha')})}, \tag{11}$$

where $\phi(\alpha) = i(1 - e^{i\lambda\alpha})/\lambda\alpha$. Also in this case, as in the Heisenberg-group case, the identification of the element $\alpha, \beta \in \mathbb{R}^2$ with the element $e^{T(\alpha,\beta)}$ makes $\mathbb{R}^2$ into a group.

We notice that the elements $e^{-ik\hat{\mathbf{x}}}e^{iw\hat{\mathbf{x}}_0}$ which appear in (6) are elements of a unitary κ-Minkowski group with $\alpha = iw$ and $\beta = -ik/\phi(iw)$, in fact

$$e^{T(\alpha,\beta)} = e^{T(0,\,\phi(\alpha)\beta)}e^{T(\alpha,0)} = e^{-ik\hat{\mathbf{x}}}e^{iw\hat{\mathbf{x}}_0}$$

with the following group law for (w, k')

$$(w, k) \cdot (w', k') = (w + w', k + e^{-\lambda w}k') \tag{12}$$

This relation is easily generalized to 1+3 dimension

$$(w, k_j) \cdot (w', k'_j) = (w + w', k_j + e^{-\lambda w}k'_j), \quad j = 1, 2, 3 \tag{13}$$

Thus we have found that the "plane waves" $\{e^{-ik_j\hat{\mathbf{x}}_j}e^{iw\hat{\mathbf{x}}_0},\ w, k_j \in \mathbb{R}\}$ in (6) are the elements of the unitary Lie group for the κ-Minkowski Lie algebra $\mathfrak{g}_\lambda$ with composition law (13) and identity given by the element $(0, 0)$. We shall refer to this group as the κ-Minkowski-group and we shall denote it with the symbol $\mathcal{U}_\lambda$.

4 Representation Theory

In this section we look for irreducible representations of the κ-Minkowski group $\mathcal{U}_\lambda$ on Hilbert spaces.

Let us remind that a representation of a group G on a Hilbert space $\mathcal{H}$ is a map $\rho :\ G \to L(\mathcal{H})$ of G into a set of linear operators on $\mathcal{H}$ satisfying the conditions

$$\begin{aligned} \rho(ab) &= \rho(a)\rho(b), \quad a, b \in G \\ \rho(e) &= \mathbf{1} \end{aligned}$$

where $\{e\}$ is the identity of G. The dimension of the representation is defined as the dimension of $\mathcal{H}$.

First of all we notice that $\mathcal{U}_\lambda$ is a solvable group. For any two elements $x, y \in G$ the set of elements $q = xyx^{-1}y^{-1}$ form a group Q called commutant of G. Set $Q \equiv Q_0$ and Q_1 the commutant of Q_0, Q_2 the commutant of Q_1 etc... If for some m we have $Q_m = \{e\}$ then the group G is said solvable. It is easy to check from (13) that in the case of $\mathcal{U}_\lambda$ the second commutant is $Q_1 = (0, 0)$, then $\mathcal{U}_\lambda$ is solvable.

We can then apply Lie's theorem which states the following (see [27]):
Lie Theorem: *Every finite-dimensional irreducible representation of a connected topological, solvable group G is one-dimensional.*

It is easy to see that the only one 1-dimensional irreducible representation of $\mathcal{U}_\lambda$ is realized for $\lambda = 0$ and it coincides with the representation of the Abelian group on the real line $\mathbb{R}$. However, we are interested in irreducible representations of κ-Minkowski for $\lambda \neq 0$ then we need to search for them in infinite dimensional Hilbert spaces.

In the first part of this Section, we show how to obtain some irreducible representations of $\mathcal{U}_\lambda$ using the JS maps. In particular, we give the explicit expression of two classes of Schrödinger representation of $\mathcal{U}_\lambda$ on $L^2(\mathbb{R})$. In the second part we use the technique of induced representations to prove that all the possible irreducible representations of $\mathcal{U}_\lambda$ are unitarily equivalent to these two classes of Schrödinger representations.

4.1 Construction of κ-Minkowski representations through Jordan-Schwinger maps

Here we show that the representation of the generators Q_j, P_j of the Heisenberg algebra provides us an easy way of obtaining representations of the κ-Minkowski generators $\hat{\mathbf{x}}_\mu$. Our idea is based on the introduction of Jordan-Schwinger maps π from the coordinates $\hat{\mathbf{x}}_\mu$ of κ-Minkowski to the generators Q_j, P_j

$$\hat{\mathbf{x}}_\mu = \pi^{-1}(Q_j, P_j)$$

These maps can be extended to the exponential functions $e^{-ik_j\hat{\mathbf{x}}_j}e^{iw\hat{\mathbf{x}}_0}$ obtaining the representations of the κ-Minkowski group on the Hilbert spaces on which the Heisenberg algebra is represented.

The JS map was originally introduced by Schwinger to deal with angular momentum in terms of harmonic-oscillator creation and annihilation operators (a_j, a_j^*), $j = 1, 2$, satisfying the standard commutation relations $[a_j, a_k^*] = \delta_{jk}$. However, since the creation and annihilation operators (a, a^*) can be connected to the Quantum Mechanics generators ($[Q, P] = i$) by the following relations

$$Q = \frac{a + a^*}{\sqrt{2}}, \quad P = \frac{a - a^*}{i\sqrt{2}}$$

we prefer to write the JS maps in terms of (Q, P).

A realization of $SL(2)$ in terms of Q, P is given by:

$$J_1 = \frac{1}{2}(Q_1Q_2 + P_1P_2) \quad J_2 = \frac{1}{2}(Q_1P_2 - Q_2P_1) \quad J_3 = \frac{1}{4}(Q_1^2 - Q_2^2 + P_1^2 - P_2^2)$$

we have in fact

$$[J_j, J_k] = i\epsilon_{jkl}J_l$$

In Ref. [22] a generalization of JS map is given for all three dimensional Lie-algebras.

The following form of the JS map π_+ is suggested for the generators of $1+1$-dimensional κ-Minkowski Lie algebra

$$\pi_+(\hat{\mathbf{x}}_0) = \lambda^2 P, \qquad \pi_+(\hat{\mathbf{x}}) = \lambda e^{-\frac{Q}{\lambda}} \tag{14}$$

This map is easily extendable to the $1+3$-dimensional case but we prefer to work in two dimensions and then to extend the result to $1+3$ dimensions.

The JS map above is a Lie-algebra homomorphism

$$[\pi_+(\hat{\mathbf{x}}_0), \pi_+(\hat{\mathbf{x}})] = \lambda^3[P, e^{-\frac{Q}{\lambda}}] = i\lambda^2 e^{-\frac{Q}{\lambda}} = i\lambda\pi_+(\hat{\mathbf{x}}) \tag{15}$$

and in particular, it is a Lie-algebra *-homomorphism

$$[\pi_+(\hat{\mathbf{x}}_\mu)]^* = \pi_+(\hat{\mathbf{x}}_\mu) = \pi_+(\hat{\mathbf{x}}_\mu^*) \tag{16}$$

Thus this map reflects the hermitian property of $\hat{\mathbf{x}}_\mu$.

This map is not unique. A wide class of JS maps can be constructed from (14) by canonical transformations

$$Q \to Q',\ P \to P' : \quad [Q', P'] = i$$

In fact, the canonical condition $[Q', P'] = i$ assures that $[\pi'(\hat{\mathbf{x}}_0), \pi'(\hat{\mathbf{x}})] = i\lambda\pi'(\hat{\mathbf{x}})$.

However another class of JS maps exists which is not possible to connect with the previous one. In fact consider the map

$$\pi_-(\hat{\mathbf{x}}_0) = \lambda^2 P, \quad \pi_-(\hat{\mathbf{x}}) = -\lambda\, e^{-\frac{Q}{\lambda}} \tag{17}$$

which also reproduces the κ-Minkowski commutation relations. Despite the two maps differ by a sign, it is not possible to find a canonical transformation $Q, P \to Q', P'$ that maps $\pi_-(\hat{\mathbf{x}})$ into $\pi_+(\hat{\mathbf{x}})$. As we shall see later, this fact has implications in the classification of the irreducible representations of the κ-Minkowski group.

The JS maps allow us to represent the operators $\hat{\mathbf{x}}_\mu$ on the Hilbert spaces in which the generators Q, P of the Heisenberg algebra are represented.

Here we focus on the Schrödinger representation of the Heisenberg algebra, without worrying about other representations. The Stone-Von Neumann theorem states in fact that once we have chosen a non-zero Plancks constant, the irreducible representation of the Heisenberg algebra is unique (see [24]). The Hilbert representation space of the Schrödinger representation is $L^2(\mathbb{R})$, the vector space of all square integrable complex-valued functions on $\mathbb{R}$,

$$L^2(\mathbb{R}) = \{\psi : \mathbb{R} \to \mathbb{C},\ \ ||\psi||^2 = \int_{\mathbb{R}} dx\, \overline{\psi(x)}\psi(x) < \infty\} \tag{18}$$

The generators Q, P are represented on $L^2(\mathbb{R})$ in the following way (see Appendix C for details about our notation):

$$Q\psi(x) = x\psi(x) \qquad P\psi(x) = -i\partial_x\psi(x)$$

Through the map (14) we get the Schrödinger representation of $\hat{\mathbf{x}}_\mu$ on $L^2(\mathbb{R})$

$$\begin{aligned} \hat{\mathbf{x}}_0\, \psi(x) &= -i\lambda^2\partial_x\, \psi(x) \\ \hat{\mathbf{x}}\, \psi(x) &= \lambda e^{-\frac{x}{\lambda}}\, \psi(x) \end{aligned}$$

where $\psi(x) \in L^2(\mathbb{R})$.

If we extend the map π_+ as *-homomorphism to the Universal Enveloping Algebra (UEA) of κ-Minkowski Lie-algebra $\mathfrak{g}_\lambda$ such that

$$\pi_+(\hat{\mathbf{x}}_0^m \hat{\mathbf{x}}^n) = \pi_+(\hat{\mathbf{x}}_0)^m \pi_+(\hat{\mathbf{x}})^n$$

we obtain the following map π_+ on the elements of the κ-Minkowski group $\mathcal{U}_\lambda$

$$\pi_+(e^{-ik\hat{\mathbf{x}}}e^{iw\hat{\mathbf{x}}_0}) = e^{-ik\pi_+(\hat{\mathbf{x}})}e^{iw\pi_+(\hat{\mathbf{x}}_0)} = e^{-i\lambda k e^{-\frac{Q}{\lambda}}}e^{i\lambda^2 wP} \tag{19}$$

By using the Quantum Mechanics formalism summarized in Appendix C, we can represent the operator π_+ on $L^2(\mathbb{R})$ and obtain

$$e^{-i\lambda k e^{-\frac{Q}{\lambda}}} e^{i\lambda^2 wP}\psi(x) \quad = \quad e^{-i\lambda k e^{-\frac{x}{\lambda}}}\psi(x+\lambda^2 w)$$

In this way we have obtained a map $\rho_+ : \ \mathcal{U}_\lambda \to U(L^2(\mathbb{R}))$ from the κ-Minkowski group $\mathcal{U}_\lambda$ to unitary operators on $L^2(\mathbb{R})$

$$\rho_+(w,k)\psi(x) = e^{-i\lambda k e^{-\frac{x}{\lambda}}}\psi(x+\lambda^2 w) \tag{20}$$

The same procedure for the map π_- (17) leads to the following unitary representation of $\mathcal{U}_\lambda$ on $L^2(\mathbb{R})$

$$\rho_-(w,k)\psi(x) = e^{i\lambda k e^{-\frac{x}{\lambda}}}\psi(x+\lambda^2 w) \tag{21}$$

At this level of discussion we have only exhibited two kinds of representations of $\mathcal{U}_\lambda$ and we have shown how they can be obtained through JS maps. We do not want to analyze here the problem of classification of all representations of the κ-Minkowski group. In the next section, however, we shall prove that any irreducible representation ρ can be connected by a unitary transformation U to the representation ρ_+ or ρ_- which exhaust the classification of all irreducible representations of the κ-Minkowski group.

It is easy to prove that these representations are irreducible. We show it in the case of ρ_+. Suppose that it exists a subspace $V \subset L^2(R)$ left invariant by $\rho_+(w,k)$. Consider a nonzero vector $\psi \in V$ and a vector $\phi \in V_\perp$ where $V_\perp$ is orthogonal to V. Since $\rho_+(w,k)\psi \in V$, we have that

$$\langle \phi, \rho_+(w,k)\psi\rangle = 0, \quad \forall w,k \in \mathbb{R}$$

and using the explicit representation of $\rho_+(w,k)$

$$\int dx\, e^{-i\lambda k e^{-\frac{x}{\lambda}}}\overline{\phi(x)}\psi(x+\lambda^2 w) = 0 \quad \forall w,k \in \mathbb{R}$$

which can be rewritten as

$$-\lambda \int dy \frac{\theta(y)}{y}\, e^{-iky}\overline{\phi(\lambda \log(\frac{\lambda}{y}))}\psi(\lambda\log(\frac{\lambda e^{\lambda w}}{y})) = 0 \quad \forall w,k \in \mathbb{R}$$

which means that

$$\frac{\theta(y)}{y}\overline{\phi(\lambda \log(\frac{\lambda}{y}))}\psi(\lambda\log(\frac{\lambda e^{\lambda w}}{y})) = 0 \quad \forall w \in \mathbb{R}, \forall y \in \mathbb{R}_+$$

setting $y = \lambda e^{-\lambda^{-1}x}$ we get

$$\overline{\phi(x)}\psi(x+\lambda^2 w) = 0 \quad \forall w \in \mathbb{R}, \forall x \in \mathbb{R}$$

Since $\psi \neq 0$ and w is arbitrary then $\phi = 0$ and $V \equiv L^2(\mathbb{R})$.

4.2 Construction of κ-Minkowski representations through the technique of the induced representations

As we will show below the κ-Minkowski group (12) is a semidirect-product group and in the case of semidirect products the technique of induced representations is very powerful. In fact there is a Theorem [27] stating that every IUR of a semidirect product G is induced from the representation of a proper subgroup of G. Thus, this technique allows us to classify all IURs representations of the group.

We say that a group G is the semidirect product of two proper subgroups $S, N \subset G$, and we denote it as $G = S \ltimes N$, if
i) the subgroup N is normal, i.e. for any $g \in G$ and $n \in N$ the element gng^{-1} is still in N.
ii) every element $g \in G$ can be written in one and only one way as $g = ns$, where $n \in N$ and $s \in S$.

The group S acts on N by conjugation. Let $s \triangleright n$ denotes the action of $s \in S$ on $n \in N$:

$$s \triangleright n = sns^{-1} \tag{22}$$

We can write the elements of G as ordered pairs $g = (s, n)$, and imagine that the subgroups $S = (s, 0)$ and $N = (0, n)$. The composition law of the semidirect product $G = S \ltimes N$ takes the following form

$$(s_1, n_1)(s_2, n_2) = (s_1 \cdot s_2,\ n_1 \cdot s_1 \triangleright n_2) \tag{23}$$

where the symbol $\cdot$ denotes the composition law in S and N.

In the case of the κ-Minkowski group $\mathcal{U}_\lambda$ (12), we can identify the groups S and N as the two Abelian subgroups $S = \{(w, 0),\ w \in \mathbb{R}\}$ and $N = \{(0, k),\ k \in \mathbb{R}\}$. The composition law $\cdot$ is the standard sum. The action of S on N is given by

$$s \triangleright n = sns^{-1} = (w, 0)(0, k)(-w, 0) = (0, e^{-\lambda w} k) \tag{24}$$

Identifying the element $(w, 0) \equiv w$ and $(0, k) \equiv k$ we write the action above as

$$w \triangleright k = e^{-\lambda w} k \tag{25}$$

The construction of induced representations of a semidirect product $S \ltimes N$ is based on two notions:

- the <u>orbits</u> $\hat{O}_{\hat{n}}$ which are the set of all points in $\hat{N}$ (the dual space of N) connected to a point $\hat{n} \in \hat{N}$ by the action of S on $\hat{n}$.
- the <u>stability group</u> $S_{\hat{O}_{\hat{n}}}$ which is a subgroup of S which leaves invariant the orbit $\hat{O}_{\hat{n}}$ ($\hat{k} \triangleleft w = \hat{k}$, for any $w \in S_{\hat{O}_{\hat{n}}}$, $\hat{k} \in \hat{O}_{\hat{n}}$).

By definition, the dual space of a group is the set of equivalence classes of all continuous, irreducible unitary representations of the group. Since in the κ-Minkowski case N is Abelian, any irreducible representation of it is one-dimensional, then it is represented in the complex space $\mathbb{C}$. The unitary irreducible representation of N consists then of the functions $\hat{n} : N \to \mathbb{C}$ such that

$$\begin{aligned} |\hat{n}(n)| &= 1 \\ \hat{n}(n_1 n_2) &= \hat{n}(n_1)\hat{n}(n_2) \end{aligned}$$

Notice that this is just the definition of characters of N, thus the space $\hat{N}$ coincides with all characters of N. In our case $N = \{(0,k)\} \sim \{k\}$ and we denote with $\hat{k} \in \hat{N}$ the dual of $k \in N$. As N is isomorphic to $\mathbb{R}$, every character $\hat{k}(\cdot)$ has the form

$$\hat{k}(k) = e^{i\hat{k}k}, \quad \hat{k} \in \mathbb{R} \tag{26}$$

and also $\hat{N}$ is isomorphic with $\mathbb{R}$ (and with N itself).

Using the duality between N and $\hat{N}$ we find the action of S on $\hat{N}$

$$\hat{k}(w \triangleright k) = e^{i\hat{k}(w\triangleright k)} = e^{i\hat{k}e^{-\lambda w}k} = e^{i(\hat{k}e^{-\lambda w})k} = (\hat{k} \triangleleft w)(k) \tag{27}$$

Thus the action of an element $w \in S$ on $\hat{k} \in \hat{N}$ is given by

$$\hat{k} \triangleleft w = e^{-\lambda w}\hat{k} \tag{28}$$

The set of all $\hat{k} \triangleleft w$ for a given $\hat{k} \in \hat{N}$ and $\forall w \in S$, is called orbit of the character $\hat{k}$, and is denoted by $\hat{O}_{\hat{k}}$. Two orbits $\hat{O}_{\hat{k_1}}$ and $\hat{O}_{\hat{k_2}}$ either coincide or are disjoint. Thus the dual space $\hat{N}$ decomposes into nonintersecting sets.

In our case, the orbit $\hat{O}_{\hat{k}}$ of the point $\hat{k} \in \hat{N}$ is the set $\hat{O}_{\hat{k}} = e^{-\lambda w}\hat{k}$ for all $w \in \mathbb{R}$. We can identify each orbit with its value $\hat{k} = l$ at $w = 0$. Thus, we can distinguish two types of orbits:

1. the orbits with $l \neq 0$
2. and the orbits with $l = 0$

The first set of orbits does not have any stability group $S_{\hat{O}} = \{0\} \subset S$, while the second orbit is a fixed point under the action of any element of S, then $S_{\hat{O}} = S$.

In order to construct and classify the IUR of κ-Minkowski we use the following theorem which states that every irreducible representation of a group G of semidirect-product type is induced from the representation of a subgroup $K \subset G$.

IUR Theorem. *Let G be a regular*[5] *, semidirect product $S \ltimes N$ of separable, locally compact groups S and N, and let N be Abelian. Let T be an IUR of G. Thus:*

i) One can associate with T an orbit $\hat{O}$ in $\hat{N}$, the dual space of N. Each orbit has a stability group $S_{\hat{O}}$

ii) The representation T is unitarily equivalent to a representation ρ^L induced by L, the irreducible representation of the group $S_{\hat{O}} \ltimes N$.

iii) The representation ρ^L in ii) is irreducible.

(for a proof see [27])

According to this Theorem, every IUR of the κ-Minkowski group (12) is a representation induced by an IUR of the stability subgroup $S_O \ltimes N$ associated with the orbits $l \neq 0$ or $l = 0$.

In Appendix A the explicit form of the representation $\rho^L_{(w,k)}$ of the element $(w,k) \in \mathcal{U}_\lambda$ is constructed:

[5] We say that G is a regular semidirect product of N and S if $\hat{N}$ contains a countable family $Z_1, Z_2, ...$ of Borel subsets, each a union of G orbits, such that every orbit in $\hat{N}$ is the intersection of the members of a suitable family $Z_{n_1}, Z_{n_2}, ...$ containing that orbit. One can show that the regularity condition is fulfilled by $\mathcal{U}_\lambda$.

1. Case $l \neq 0$. The representation is induced by the representation L of $\{0\} \ltimes S = S \sim \mathbb{R}$, and takes the form

$$\rho^L_{(w,k)}\psi(x) = e^{ikle^{-\frac{x}{\lambda}}}\psi(x+\lambda^2 w) \tag{29}$$

where $\psi \in L^2(\mathbb{R};\mathbb{C})$. This representation is parametrized by the parameter $l \neq 0$.

2. Case $l = 0$. The representation is induced by the representation L of $S \ltimes N = G$ and corresponds to the one-dimensional representation, that is the character

$$\rho^L_{(w,k)} = e^{iwc}I, \quad c \in \mathbb{R} \tag{30}$$

We discuss case 1 in more detail.
The parameter l has the dimension of a length and we can write it using the natural length scale of our model *i.e.* the noncommutativity parameter λ, and since $l \neq 0$, we write

$$l = \pm e^{\lambda^{-1}\alpha}\lambda \quad \alpha \in \mathbb{R} \tag{31}$$

We split the discussion of case 1 into two subcases characterized by the positive and negative signs of the parameter l. Let us focus on the first subcase (l¿0)

$$\rho^\alpha_{(w,k)}\psi(x) = e^{i\lambda k e^{\lambda^{-1}(\alpha - x)}}\psi(x+\lambda^2 w)$$

Notice that the case $\alpha = 0$ reproduces the Schrödinger representation $\rho_-(w,k)$ obtained with the JS map (17)

$$\rho^{\alpha=0}_{(w,k)}\psi(x) = \rho_-(w,k)\psi(x)$$

Let us introduce now the unitary transformation

$$U\psi(x) = \psi(x+\alpha) \tag{32}$$

We see that

$$\begin{aligned} U\rho_-(w,k)U^*\psi(x) &= U\rho_-(w,k)\psi(x-\alpha) = Ue^{i\lambda k e^{-\lambda^{-1}(x-\alpha)}}\psi(x-\alpha+\lambda^2 w) \\ &= e^{i\lambda k e^{\lambda^{-1}(\alpha - x)}}\psi(x+\lambda^2 w) = \rho^\alpha_{(w,k)}\psi(x) \end{aligned} \tag{33}$$

This result shows that any representation $\rho^\alpha_(w,k)$ is unitarily equivalent to the Schrödinger representation $\rho_-(w,k)$ obtained with the JS map (17).

The same thing happens for the second subcase where the representations with different α are all unitarily equivalent to the representation $\rho_+(w,k)$ obtained with the JS map (14).

At the end of the day we found that for the case $l \neq 0$ (which is the one we are interested in) the irreducible representations of the κ-Minkowski group are all unitarily equivalent to the two inequivalent representations $\rho_+(w,k)$ and $\rho_-(w,k)$ obtained with the JS maps $\pi_\pm$

$$U^*\rho^{(\alpha,\pm)}_{(w,k)}U = \rho_\mp(w,k) \tag{34}$$

5 C*-ALGEBRAS

5.1 C*-algebra of Weyl operators

As we mentioned in the Introduction, the Weyl quantization of a function $f(q,p)$ of the classical phase space $\mathbb{R}^2$ is the linear map

$$W(f(q,p)) = \int d\alpha d\beta \tilde{f}(\alpha,\beta) e^{\alpha Q + i\beta P}$$

where $\tilde{f}$ is the Fourier transform of f, and $\tilde{f} \in L^1(\mathbb{R}^2)$. Using the Schrödinger representation of Q, P , $W(f)$ can be represented on $L^2(\mathbb{R})$ as follows

$$W(f)\psi(x) \quad = \quad \int d^n\alpha d^n\beta \tilde{f}(\alpha,\beta) e^{i\alpha x} e^{i\alpha\beta/2} \psi(x+\beta) \tag{35}$$

This can be written by introducing the kernel $K(x,y)$ of the operator $W(f)$

$$W(f)\psi(x) \quad = \quad \int dy K(x,y)\psi(y)$$

with the kernel given by

$$K(x,y) = \int d\alpha\, \tilde{f}(\alpha, y-x)\, e^{i\alpha(x+y)/2}$$

Since $\tilde{f} \in L^1(\mathbb{R}^2)$, then the operators $W(f)$ are bounded operators on $L^2(\mathbb{R})$

$$||W(f)|| \leq ||\tilde{f}||_1$$

The set all bounded operators $W(f)$ on $L^2(\mathbb{R})$ equipped with the standard product, the norm

$$||W|| = \sup\{\frac{||W(f)\psi(x)||_2}{||\psi||_2} : \psi \in L^2(\mathbb{R})\}$$

and the involution given by the adjoint operation on $L^2(\mathbb{R})$, is a C^*-algebra. We show now that the C^*-algebra generated by $W(f)$ is a C^*-algebra of compact operators on $L^2(\mathbb{R})$.

Theorem: *The operator $W(f)$ defined by (35) is a compact operator on $L^2(\mathbb{R})$ for all $\tilde{f} \in L^1(\mathbb{R}^2)$.*

Proof:
Let us show first that if $\tilde{f} \in L^2(\mathbb{R}^2)$ then $W(f)$ is the set of Hilbert-Schmidt operators, *i.e.*

$$tr[W^*(f)W(f)] < \infty$$

The trace can be computed with the kernel in the following way

$$\begin{aligned} tr[W^*(f)W(f)] &= \int dxdy\, K_{W(f)}(x,y)\, K_{W^*(f)}(y,x) \\ &= \int dxdy\, K_{W(f)}(x,y)\, \overline{K_{W(f)}(x,y)} = \int dxdy\, |K_{W(f)}(x,y)|^2 \end{aligned}$$

Since the map $f \to K_f$ is unitary, hence norm preserving, we have that

$$\int dxdy\,|K_{W(f)}(x,y)|^2 = \int d\alpha d\beta|\tilde{f}(\alpha,\beta)|^2 < \infty \quad \forall \tilde{f} \in L^2(\mathbb{R}^2)$$

Thus, $W(f)$ is compact for $f \in L^1 \cap L^2$, hence for $f \in L^1$ since $||W(f)|| \leq ||f||_1$ and the norm limit of compact operators is compact [24].

5.2 C^*-algebra of κ-Minkowski operators

The linear map (6)

$$\mathcal{W}(f) = \int d^2k\ \tilde{f}(k)\, e^{-ik\hat{\mathbf{x}}} e^{iw\hat{\mathbf{x}}_0} \tag{36}$$

can be represented on $L^2(\mathbb{R})$ using the representation (29) (setting $l = \lambda$ to be simple)

$$\mathcal{W}(f)\psi(x) \quad = \quad \int d^2k\ \tilde{f}(k)\, e^{i\lambda k e^{-\frac{x}{\lambda}}} \psi(x + \lambda^2 w) \tag{37}$$

Since $\tilde{f} \in L^1(\mathbb{R}^2)$ the operators $\mathcal{W}(f)$ are bounded operators on $L^2(\mathbb{R})$

$$||\mathcal{W}(f)|| \leq ||\tilde{f}||_1 \tag{38}$$

The set all bounded operators $\mathcal{W}(f)$ on $L^2(\mathbb{R})$ equipped with the standard product, the norm

$$||\mathcal{W}|| = \sup\{\frac{||\mathcal{W}\psi(x)||}{||\psi||} : \psi \in L^2(\mathbb{R})\} \tag{39}$$

and the involution given by the adjoint operation on $L^2(\mathbb{R})$, is a C^*-algebra.

Now we ask under which hypotheses the κ-Minkowski operators $\mathcal{W}(f)$ are compact operators. Following the case of Weyl operators $W(f)$ we compute the trace

$$\mathrm{tr}[\mathcal{W}^*(f)\mathcal{W}^*(f)] = \int dxdy|K_{\mathcal{W}(f)}(x,y)|^2 \tag{40}$$

where the kernel $K_{\mathcal{W}(f)}$ is given by

$$K_{\mathcal{W}(f)}(x,y) = \int dwdk\ \tilde{f}(w,k)\, e^{i\lambda k e^{-\frac{x}{\lambda}}} \delta(x - y + \lambda^2 w) \tag{41}$$

A direct calculation shows that

$$\begin{aligned}
\int dxdy|K_{\mathcal{W}(f)}(x,y)|^2 \quad &= \quad \int dxdy\, dwdkdw'dk'\ \tilde{f}(w,k)\overline{\tilde{f}(w',k')} e^{i\lambda k e^{-\frac{x}{\lambda}}} e^{-i\lambda k' e^{-\frac{x}{\lambda}}} \\
&\quad\ \delta(x - y + \lambda^2 w)\delta(x - y + \lambda^2 w') \\
&= \quad \lambda^{-2} \int dxdy\, dwdkdw'dk'dzdz'\ \tilde{f}(w,k)\overline{\tilde{f}(w',k')} e^{i\lambda k e^{-\frac{x}{\lambda}}} e^{-i\lambda k' e^{-\frac{x}{\lambda}}} \\
&\quad\ e^{-iz\frac{x-y}{\lambda^2}} e^{iz'\frac{x-y}{\lambda^2}} e^{-izw} e^{izw'} \\
&= \quad \lambda^{-2} \int dxdy\, dzdz'\ f(z, e^{-\frac{x}{\lambda}})\overline{f(z', e^{-\frac{x}{\lambda}})} e^{-i(z-z')\frac{x}{\lambda^2}} e^{i(z-z')\frac{y}{\lambda^2}} \\
&= \quad \int dx\, dzdz'\ f(z, e^{-\frac{x}{\lambda}})\overline{f(z', e^{-\frac{x}{\lambda}})} e^{-i(z-z')\frac{x}{\lambda^2}} \delta(z - z') \\
&= \quad \int dz\, dx\ |f(z, e^{-\frac{x}{\lambda}})|^2 = \int dzdx \frac{\lambda\theta(x)}{x}\ |f(z,x)|^2
\end{aligned}$$

Thus the κ-Minkowski operators $\mathcal{W}(f)$ are Hilbert-Schmidt operators if $f \in L^2(\mathbb{R}^2, d\mu)$ with $d\mu = \lambda dz dx \theta(x)/x$. And, as we showed for Weyl operators, $\mathcal{W}(f)$ is compact for $f \in L^1 \cap L^2$, hence for $f \in L^1(\mathbb{R}^2, d\mu)$ since $||\mathcal{W}(f)|| \leq ||f||_1$ and the norm limit of compact operators is compact.

6 Construction of a cyclic integration

In this section we focus on one particular choice of representation of $\hat{\mathbf{x}}_\mu$ among the ones illustrated above

$$\begin{aligned} \pi(\hat{\mathbf{x}}_0)\psi(x) &= -\frac{\lambda}{2}\sum_{j=1}^{3}[Q_jP_j + P_jQ_j]\psi = i\lambda\sum_{j=1}^{3}(x_j\partial_j + \frac{1}{2})\psi \\ \pi(\hat{\mathbf{x}}_j)\psi(x) &= Q_j\psi(x) = x_j\psi(x) \end{aligned} \tag{42}$$

with the factor λ for the correct physical dimensions.

The JS map (42) is connected with the JS maps $\pi_\pm$ through the transformation

$$Q \to Q' = \pm\lambda e^{-\frac{Q}{\lambda}} \qquad P \to P' = \mp(\lambda P - \frac{i}{2})e^{\frac{Q}{\lambda}} \tag{43}$$

We notice that the transformation $Q \to e^{-\frac{Q}{\lambda}}$ maps the Hermitian operator Q into a positive operator. The other transformation $Q \to -e^{-\frac{Q}{\lambda}}$ instead maps Q into a negative operator. Thus the JS map π is connected to π_+ for the spectrum $x > 0$ and with π_- for the spectrum $x < 0$.

We show in Appendix B that the two representations ρ and ρ_+ of the κ-Minkowski group, obtained from the JS maps π and π_+, are unitarily equivalent, i.e., there is a unitary transformation $U: \ L^2(\mathbb{R}) \to L^2(\mathbb{R}_+)$, such that

$$\rho(w,k) = U\rho_+(w,k)U^* \tag{44}$$

Now we use the representation in Eq. (42) of the κ-Minkowski algebra in order to obtain a rule of integration in κ-Minkowski noncommutative spacetime.

Let us remind that the trace of the operators $\mathcal{W}(f)$ is defined as

$$\operatorname{tr}\mathcal{W}(f) = \int dx K(x,x) \tag{45}$$

where $K(x,x)$ is the kernel of $\mathcal{W}(f)$

$$\mathcal{W}(f)\psi(x) = \int dy K(x,y)\psi(y)$$

It easy to see that for the representation (42)

$$K(x,x) = \int dwdk\tilde{f}(w,k)e^{-\frac{\lambda w}{2}}e^{-ikx}\delta(e^{-\lambda w}x - x) = \int dwdk\frac{1}{\lambda|x|}\tilde{f}(w,k)e^{-\frac{\lambda w}{2}}e^{-ikx}\delta(w)$$

and thus the trace is

$$\operatorname{tr}\mathcal{W}(f)=\int dxdk\frac{1}{\lambda|x|}\tilde{f}(0,k)e^{-ikx}$$

This result can be easily extended to $3+1$ dimensions

$$\operatorname{tr}\mathcal{W}(f)\quad=\quad\int\frac{d^3x}{\lambda^3|\vec{x}|^3}d^3k\;\tilde{f}(0,\vec{k})e^{-i\vec{k}\vec{x}}$$

and, using the Fourier transform formulas, we obtain the final result:

$$\operatorname{tr}\mathcal{W}(f)=\int\frac{dx_0d^3x}{\lambda^3|\vec{x}|^3}f(x_0,\vec{x}) \tag{46}$$

In this way we can see that the trace (45) of the κ-Minkowski operators $\mathcal{W}(f)$ corresponds to the cyclic integral on κ-Minkowski spacetime obtained in [18] and [28]. In these papers, where the star-product $(*)$ formalism was used, a cyclic integral was constructed by introducing a non-trivial measure

$$I[\mathcal{W}(f)\mathcal{W}(g)]\equiv\int d^4x\,\mu(x)\,(f(x)*g(x)) \tag{47}$$

and, in particular, in [18] the explicit expression $\mu(x)=\frac{1}{|\vec{x}|^3}$ was obtained by imposing the cyclicity condition

$$\int d^4x\,\mu(x)\,[f*g-g*f]=0 \tag{48}$$

Of course, this condition is naturally fulfilled by the request that the integral in κ-Minkowski correspond to a trace of operators on Hilbert space

$$\operatorname{tr}[\mathcal{W}(f)\mathcal{W}(g)]=\operatorname{tr}[\mathcal{W}(g)\mathcal{W}(f)] \tag{49}$$

7 Summary and Outlook

The Weyl formulation of Quantum Mechanics is based on the introduction of (Weyl) operators. Looking for a formulation of a field theory in κ-Minkowski noncommutative spacetime a generalization of Weyl operators has been introduced. In this paper we studied the problem of the representations of these κ-generalized Weyl operators on Hilbert spaces. They are essentially Fourier expansions in terms of elements of the Lie group of κ-Minkowski. Therefore it has been sufficient to study the representations of the κ-Minkowski group. Introducing Jordan-Schwinger maps between Quantum Mechanics generators and κ-Minkowski coordinates the representations of κ-Minkowski group is immediately obtained. However many JS maps, related by canonical transformations, are possible giving rise to several representations of the κ-Minkowski group. The problem of the classification of all unitary irreducible representations has been solved by the technique of induced representations, and we found that all IURs of the κ-Minkowski group are essentially led back to two types of Schrödinger representations on $L^2(R)$. As a natural application of the representation theory one can construct an integration on κ-Minkowski spacetime as trace of

operators. This integration is automatically cyclic in the arguments and reproduces the result obtained in literature using a different approach. The cyclic integration involves the presence of a nontrivial measure $d\mu$, which seems a characterization of κ-Minkowski with respect to noncommutative spaces with canonical structure (including the Quantum Mechanics phase space as well). The approach of the present paper, based on the representations of κ-Minkowski fields as operators on Hilbert space, opens a new way to study a field theory on κ-Minkowski noncommutative spacetime. As one interesting application of this approach one could construct the area and volume operators in κ-Minkowski and compute their spectra. This could provide a comparison with the results of Loop Quantum Gravity where a detailed computation of the area/volume spectra has been done.

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A Construction of induced representation

In this appendix we construct explicitly the induced representations of κ-Minkowski group.

Let L_h be a unitary representation of a closed subgroup $H \subseteq G$ in a separable Hilbert space $\mathcal{H}$. Let $d\mu(x)$ a quasi-invariant measure[6] on the homogeneous space $X = H\diagdown G = \{Hg,\ g \in G\}$ of the right H-coset. Consider the set $\mathcal{H}^L$ of all function $u:\ G \to \mathcal{H}$ such that:

1. $\langle u(g), v\rangle$ is measurable for all $v \in \mathcal{H}$.
2. $u(hg) = L_h u(g)$, for all $h \in H$ and all $g \in G$.
3. $\int_X \|u(g)\|^2_{\mathcal{H}}\, d\mu(x) < \infty$.

The map $g_0 \to \rho^L_{g_0}$ given by

$$\rho^L_{g_0} u(g) = \sqrt{\frac{d\mu(xg_0)}{d\mu(x)}}\, u(gg_0) \tag{50}$$

defines a unitary representation of G in $\mathcal{H}^L$.
Proof: It easy to see that the map is a representation of G in $\mathcal{H}^L$

$$\rho^L_{g_1}\rho^L_{g_0}u(g) = \sqrt{\frac{d\mu(xg_0)}{d\mu(x)}}\rho^L_{g_1}u(gg_0) = \sqrt{\frac{d\mu(xg_0g_1)}{d\mu(x)}}u(gg_0g_1) = \rho^L_{g_0g_1}u(g)$$

The unitarity is simply proved by noticing that

$$\int_X \|\rho^L_{g_0}u(g)\|^2 d\mu(x) = \int_X \|u(gg_0)\|^2 d\mu(xg_0) = \int_X \|u(g)\|^2 d\mu(x)$$

[6] A quasi-invariant measure $d\mu(x)$ on $X = G \setminus H$ is a positive measure such that $d\mu(xg) = \rho(x)d\mu(x)$ for any $g \in G$. The function $\rho(x)$ is said Radom-Nikodym derivative.

If we consider $H = S_{\hat{O}} \ltimes N$, where $S_{\hat{O}}$ is the stability group of the orbit $\hat{O}$, the formula (50) can be rewritten making use of the following Lemmas.

Lemma 1. Every IUR L of $S_{\hat{O}} \ltimes N$ is determined and determines an IUR $L_{S_{\hat{O}}}$ of $S_{\hat{O}}$

$$L_{(s_{\hat{O}},n)} = \langle n, \hat{n}_f \rangle L_{s_{\hat{O}}},$$

where $(s_{\hat{O}}, n) \in H = S_{\hat{O}} \ltimes N$, and $\hat{n}_f$ is the fixed point under the action of $S_{\hat{O}}$ ($\hat{n}_f \lhd s_{\hat{O}} = \hat{n}_f$ for any $s_{\hat{O}} \in S_{\hat{O}}$.
Proof: it is easy to see that

$$\begin{aligned}
L_{(s_{\hat{O}},n)(s'_{\hat{O}},n')} &= L_{(s_{\hat{O}}s'_{\hat{O}},n+s'_{\hat{O}}\rhd n')} = \langle n + s'_{\hat{O}} \rhd n', \hat{n}_f \rangle L_{s_{\hat{O}}s'_{\hat{O}}} \\
&= \langle n, \hat{n}_f \rangle \langle s'_{\hat{O}} \rhd n', \hat{n}_f \rangle L_{s_{\hat{O}}} L_{s'_{\hat{O}}} = \langle n, \hat{n}_f \rangle L_{s_{\hat{O}}} \langle n', \hat{n}_f \rangle L_{s'_{\hat{O}}} \\
&= L_{(s_{\hat{O}},n)} L_{(s'_{\hat{O}},n')}
\end{aligned}$$

and

$$L^*_{(s_{\hat{O}},n)} = \overline{\langle n, \hat{n}_f \rangle} L^{-1}_{s_{\hat{O}}} = \langle n^{-1}, \hat{n} \rangle = L^{-1}_{(s_{\hat{O}},n)}$$

Lemma 2. The set of function $u \in \mathcal{H}^L$, such that $u(hg) = L_h u(g)$ for any $h \in H = S_{\hat{O}} \ltimes N$, can be written as

$$u(g) = \langle n, \hat{n} \rangle u(s), \quad g = (s, n) \tag{51}$$

with the condition $u(s_{\hat{O}}s) = L_{s_{\hat{O}}} u(s)$ for any $s_{\hat{O}} \in S_{\hat{O}}$.
Proof: Let $h = (s'_{\hat{O}}, n')$ then a simple calculation shows that

$$\begin{aligned}
u(hg) &= u[(s'_{\hat{O}}, n')(s, n)] = u[s'_{\hat{O}}s,\ n' + s'_{\hat{O}} \rhd n] \\
&= \langle n' + s'_{\hat{O}} \rhd n, \hat{n}_f \rangle u(s_{\hat{O}}s) = \langle n', \hat{n}_f \rangle \langle s'_{\hat{O}} \rhd n, \hat{n}_f \rangle u(s'_{\hat{O}}s) \\
&= \langle k, \hat{n}_f \rangle \langle n, \hat{n}_f \rangle L_{s'_{\hat{O}}} u(s) = \langle k, \hat{n}_f \rangle L_{s'_{\hat{O}}} u(g) = L_{(s'_{\hat{O}},n')} u(g) \\
&= L_k u(g)
\end{aligned}$$

We notice also that the homogeneous space $X = [S_{\hat{O}} \ltimes N]\diagdown G$ is isomorphic to the space of the orbits $\hat{O}$

$$X = [S_{\hat{O}} \ltimes N]\diagdown G = [S_{\hat{O}} \ltimes N]\diagdown[S \ltimes N] \sim S_{\hat{O}}\diagdown S \sim \hat{O}$$

thus

$$\frac{d\mu(xg_0)}{d\mu(x)} = \frac{d\mu(ss_0)}{d\mu(s)} \tag{52}$$

Finally substituting (51) and (52) into (50) we obtain for $G \ni g_0 = (s_0, n_0)$

$$\begin{aligned}
\rho^L_{g_0} u(s) &= \langle n, \hat{n} \rangle^{-1} \sqrt{d\mu(ss_0)/d\mu(s)} \langle n + s \rhd n_0, \hat{n} \rangle u(ss_0) \\
&= \sqrt{d\mu(ss_0)/d\mu(s)} \langle s \rhd n_0, \hat{n} \rangle u(ss_0) \\
&= \sqrt{d\mu(ss_0)/d\mu(s)} \langle n_0, \hat{n} \lhd s \rangle u(ss_0)
\end{aligned} \tag{53}$$

where $\mathcal{H}^L \ni u :\ S \to \mathcal{H}$ and satisfy $u(s'_{\hat{O}}s) = L_{s'_{\hat{O}}} u(s)$, and $\hat{n} \in \hat{O}$ thus it is left fixed under the action of $S_{\hat{O}}$.

This formula gives rise to two representation according to the values of $\hat{n}$ and the correspondent stability group $S_{\hat{O}}$.

1. Case $\hat{n} = l \neq 0$. The representation is induced by the representation L of $\{0\} \ltimes S = S \sim \mathbb{R}$.
   The character $\langle n_0, \hat{n} \lhd s\rangle = e^{in_0 l e^{-\lambda s}}$. The measure $d\mu(s)$ quasi-invariant on S can be chosen as ds in fact

   $$d(ss_0) = d(s + s_0) = ds$$

   In this case we have that the factor $\sqrt{\frac{d\mu(ss_0)}{d\mu(s)}} = 1$. The (53) takes the form

   $$\rho^L_{(s_0,n_0)} u(s) = e^{in_0 l e^{-\lambda s}} u(s + s_0)$$

   where $u \in L^2(\mathbb{R}, \mu; \mathbb{C})$. This representation is parametrized by the parameter $l \neq 0$.

2. Case $\hat{n} = 0$. The representation induced by the representation L of $S \ltimes N = G$ In this case the (53) is

   $$\rho^L_{(s_0,s_0)} u(s) = u(s + s_0)$$

   and since the stability group is S, we have that $u(s + s_0) = L_{s_0} u(s)$ for any $s_0 \in S$. Thus the induced representation is the one-dimensional representation, that is the character

   $$\rho^L_{g_0} = e^{is_0 c} I, \quad c \in \mathbb{R}$$

B Example of unitary equivalence between two κ-Minkowski group representations

Here we want to show explicitly that the two representations of the κ-Minkowski group ρ and ρ_+, obtained from the JS maps π and π_+, are unitarily equivalent, i.e., there is a unitary transformation $U : L^2(\mathbb{R}) \to L^2(\mathbb{R}_+)$, such that

$$\rho(w, k) = U \rho_+(w, k) U^* \tag{54}$$

In our case the unitary transformation U is given by

$$\psi(x) \to U\psi(x) = \frac{\theta(x)}{\sqrt{\frac{x}{\lambda}}} \psi(\lambda \log \frac{\lambda}{x}) \tag{55}$$

This transformation can be easily proved to be unitary

$$\begin{aligned} ||U\psi||^2 &= \int dx |U\psi|^2 = \int_0^\infty dx \frac{\lambda}{x} | \psi(\lambda \log \frac{\lambda}{x})|^2 = \int_0^\infty d(\lambda \log \frac{\lambda}{x}) \, |\psi(\lambda \log \frac{\lambda}{x})|^2 \\ &= \int_{-\infty}^{+\infty} dz |\psi(z)|^2 = ||\psi||^2 \end{aligned}$$

Using the explicit form of the representation ρ_+

$$\begin{aligned}
\rho_+(w,k)U\psi(x) &= \frac{\theta(x)}{\sqrt{\frac{x}{\lambda}}}\rho_+(w,k)\psi(\lambda\log\frac{\lambda}{x}) = \frac{\theta(x)}{\sqrt{\frac{x}{\lambda}}}e^{-ikx}\psi(\lambda[\log\frac{\lambda}{x}+\lambda w]) \\
&= e^{-\lambda w/2}\frac{\theta(e^{-\lambda w}x)}{\sqrt{\frac{e^{-\lambda w}x}{\lambda}}}e^{-ikx}\psi(\lambda[\log\frac{\lambda}{x}+\log e^{\lambda w}]) \\
&= e^{-\lambda w/2}\frac{\theta(e^{-\lambda w}x)}{\sqrt{\frac{e^{-\lambda w}x}{\lambda}}}e^{-ikx}\psi(\lambda\log\frac{\lambda}{xe^{-\lambda w}}) \\
&= e^{-\lambda w/2}e^{-ikx}U\psi(e^{-\lambda w}x)
\end{aligned}$$

thus:

$$\rho(w,k)\psi(x) = e^{-\lambda w/2}e^{-ikx}\psi(e^{-\lambda w}x) \tag{56}$$

The unitary transformation $U\psi(x) = \frac{\theta(-x)}{\sqrt{-\frac{x}{\lambda}}}\psi(\lambda\log\frac{\lambda}{-x})$ allows us to map ρ with the other representation ρ_- in the negative real line $\mathbb{R}_-$

$$\begin{aligned}
\rho_-(w,k)U\psi(x) &= \frac{\theta(-x)}{\sqrt{\frac{|x|}{\lambda}}}\rho_-(w,k)\psi(\log\frac{\lambda}{|x|}) = \frac{\theta(-x)}{\sqrt{\frac{|x|}{\lambda}}}e^{-ikx}\psi(\lambda[\log\frac{\lambda}{|x|}+\lambda w]) \\
&= e^{-\lambda w/2}\frac{\theta(-e^{-\lambda w}x)}{\sqrt{\frac{e^{-\lambda w}|x|}{\lambda}}}e^{-ikx}\psi(\lambda[\log\frac{\lambda}{|x|}+\log e^{\lambda w}]) \\
&= e^{-\lambda w/2}\frac{\theta(-e^{-\lambda w}x)}{\sqrt{\frac{e^{-\lambda w}|x|}{\lambda}}}e^{-ikx}\psi(\lambda\log\frac{\lambda}{|x|e^{-\lambda w}}) \\
&= e^{-\lambda w/2}e^{-ikx}U\psi(e^{-\lambda w}x)
\end{aligned}$$

C Quantum Mechanics formalism

We can see the formalism used in terms of bra-ket notation (see Sakurai [29]). We suppose the element $\psi(x)$ of the Hilbert state be

$$\psi(x) =< x|\psi > \tag{57}$$

The action of the position-momentum operators over $\psi(x)$ is

$$Q\psi(x) = x\psi(x), \quad P\psi(x) = -i\partial_x\psi(x) \tag{58}$$

Let us check the commutation relation

$$QP\psi(x) =< x|QP|\psi >= x < x|P|\psi >= -ix\partial_x\psi(x)$$
$$PQ\psi(x) =< x|PQ|\psi >= -i\partial_x < x|Q|\psi >= -i\partial_x(x\psi(x)) = -i\psi(x) - ix\partial_x\psi(x)$$

Thus

$$[Q,P]\psi(x) = i\psi(x) \tag{59}$$

Finally, let us compute the action of the operator $e^{iaQ}e^{ibP}$ on the element $\psi(x)$

$$\begin{aligned} e^{iaQ}e^{ibP}\psi(x) &= < x|e^{iaQ}e^{ibP}|\psi >= e^{iax} < x|e^{ibP}|\psi > \\ &= e^{iax}e^{b\partial_x} < x|\psi >= e^{iax}\psi(x+b) \end{aligned} \tag{60}$$

This formula has been used for getting Equations (20, 21).

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